Subsonic-supersonic condition for shocks

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G. R. Fowles

Department of Physics, Washington State University, Pullman, Washington 99163
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The necessary conditions for stability of a shock, that the shock travel with supersonic velocity with respect to the medium ahead and with subsonic velocity with respect to the medium behind, are shown to be implied by the second law of thermodynamics for very general classes of viscous, heat-conducting fluids; the Weyl conditions are not invoked. The results are shown to be also compatible with the Le Chatelier–Braun principle. They further imply that under certain conditions it is not thermodynamically permissible to assume the existence of a shock transition layer in which entropy production is due to heat conduction alone.

I. INTRODUCTION

In studies of the structure and stability of shock waves it is usually demonstrated or assumed that a necessary condition for a stable shock to exist is that the shock travel with supersonic velocity with respect to the medium ahead of the wave and with subsonic velocity with respect to the medium behind.[1-3] These restrictions can be expressed by the inequalities

$$M_0 \geqq 1, \quad M_1 \leqq 1, \tag{1}$$

where

$$M = |v/c|$$

is the Mach number, given by the magnitude of the ratio of the flow velocity to the local sound speed in a coordinate system in which the shock front is stationary. Subscripts "0" refer to the initial state ahead of the shock front and subscripts "1" to the final state behind.

For restricted classes of materials conditions (1) have been shown to follow from the second law of thermodynamics. Thus, Weyl proved (1) for materials subject to the conditions

$$(\partial^2 P/\partial V^2)_S > 0, \quad (\partial P/\partial S)_V > 0. \tag{2}$$

Landau and Lifshitz also show that (2) implies (1) for compressive shocks in both the weak shock approximation, in which no distinction is made between Hugoniot and isentropic pressure-volume curves, and for shocks of arbitrary strength (Ref. 4, p. 323). Morduchow and Libby have derived (1) for an ideal gas.[5] Cowperthwaite considered a case in which (2) is violated in the neighborhood of the initial state and showed that a postulated compressive shock with final state in this region is impossible because it would result in a net entropy decrease.[6] This result is consistent with stability arguments based on acoustic wave interactions as described by Duvall.

For rarefaction shocks the arguments are reversed so that (1) is valid when $(\partial^2 P/\partial V^2)_S < 0$.

In this paper we show that inequalities (1) are a necessary consequence of the second law of thermodynamics for very general classes of fluids. In particular, we do not assume (2), but only the well-known thermodynamic stability criteria,

$$(\partial P/\partial V)_S < (\partial P/\partial V)_T < 0 \tag{3a}$$

and

$$C_p = T(\partial S/\partial T)_p > C_v = T(\partial S/\partial T)_V > 0. \tag{3b}$$

We also assume the existence of an equilibrium surface, or fundamental equation, representing the locus of reversible paths joining equilibrium states. This surface is characterized by the usual relation,

$$S = S(E, \overline{V}, N), \tag{4}$$

where S is entropy, E is internal energy, N is molenumber, and $\overline{V}$ is volume. If we denote specific quantities per unit mass by s, e, and V and consider only closed systems, then (4) is equivalent to

$$s = s(e, V)$$

which is assumed to be invertible to give $e = e(V, s)$. The equilibrium pressure and temperature are defined by

$$P = P(e, V) \equiv -(\partial e/\partial V)_s, \quad T = T(e, V) \equiv (\partial e/\partial s)_V. \tag{5}$$

We place no restriction on the amplitude of the shock, nor do we assume an explicit form for the constitutive equations relating the viscous dissipation and the heat conduction to velocity and temperature gradients.

II. THERMODYNAMIC PRINCIPLES

Truesdell has given a general formulation of the first and second laws for irreversible processes.[7] Following his development we write the first law in the form,

$$\rho \dot{e} = w + \operatorname{div} \mathbf{h}, \tag{6}$$

where ρ is density, $\dot{e}$ is the time rate of change of the specific internal energy of a mass element, w is the rate at which mechanical work is performed on an infinitesimal volume containing the mass element, and h is the negative of the heating flux vector. The heating influx is therefore,

$$q = \mathbf{h} \cdot \mathbf{n},$$

where n is the outward unit normal to the surface of the volume.

The internal dissipation is defined by the relation

$$\delta = T\dot{s} - V \operatorname{div} \mathbf{h}. \tag{7}$$

It is thus given by the difference between the rate of entropy increase, multiplied by temperature, and the rate of energy increase due to heat conduction.

For reversible processes Gibbs' relation applies

$$\dot{e} = T\dot{s} - P\dot{V}. \tag{8}$$

Hence, combining (7) and (8),

$$\delta = \dot{e} + P\dot{V} - V\,\mathrm{div}\mathrm{h}$$

and, employing (6),

$$\rho\delta = w + \rho P\dot{V}. \tag{9}$$

We now specialize to one-dimensional flow and denote the total stress acting in the direction of the flow by σ, with compressive stress measured positive. Then,

$$w = -\rho\sigma\dot{V}$$

and, therefore

$$\delta = -(\sigma - P)\dot{V}. \tag{10}$$

Note that in this derivation we have not necessarily assumed that P, T, or s, are given by their local current values, which have not been defined, but are given instead by their values on the associated equilibrium surface. The real, nonequilibrium path followed by a material element is thus mapped onto the equilibrium surface at corresponding values of e and V. These latter quantities, of course, are well-defined whether or not equilibrium obtains. In effect, we observe the "shadow" of the real process on the equilibrium surface.

From this point of view it is not obvious that the second law need always apply in terms of the quantities thus defined. For small deviations from equilibrium the relation (4) still holds, however, and the Second Law is also assumed to be valid. We shall restrict our attention to such small deviations in the following. This point of view is discussed also by Landau and Lifshitz (Ref. 4, p. 187).

The second law, requiring that entropy production be positive, is expressed by the Clausius–Duhem inequality,

$$\rho\delta - \mathrm{h}\cdot\boldsymbol{\gamma} \geq 0, \tag{11}$$

where $\boldsymbol{\gamma}$ is defined as

$$\boldsymbol{\gamma} = (-1/T)\,\mathrm{grad}\,T$$

$$= (-1/T)(\partial T/\partial x). \tag{12}$$

III. APPLICATION TO SHOCKS

We now apply these principles, expressed by relations (6)–(12), to a steady wave of "permanent regime," or shock wave. The shock transition region joins initial and final states which are assumed to be in internal thermodynamic equilibrium; it is depicted in a coordinate system in which the front is stationary in Fig. 1.

Since the wave is stationary and one-dimensional, we can write the relation between material and spatial time derivatives as

$$d/dt = \partial/\partial t + v\,\partial/\partial x = v\,\partial/\partial x. \tag{13}$$

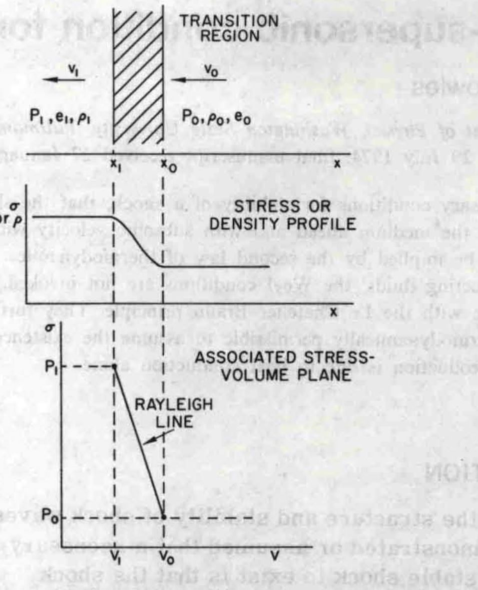

FIG. 1. Shock transition layer, compressive shock.

Consequently, (12) can be expressed as,

$$\gamma = (-1/T)(\partial T/\partial x) = -(1/Tv)(dT/dt)$$

$$= -T^{-1}v^{-1}(dT/dV)\dot{V}, \tag{14}$$

where dT/dV is the directional derivative in the equilibrium surface of temperature with respect to volume along the projection of the path followed by a material element as it traverses the shock front, and $\dot{V}$ is the material time derivative, $\dot{V} = dV/dt$.

Application of the Clausius–Duhem inequality, (11) also requires expressions for δ and for h. From (7), (10), and (13) we have

$$\mathrm{div}h = \partial h/\partial x = \rho T\dot{s} - \rho\delta$$

$$= \rho[T\dot{s} + (\sigma - P)\dot{V}] \tag{15}$$

or

$$dh = \rho v[T\,ds/dV + (\sigma - P)]dV. \tag{16}$$

To first-order terms in a series expansion about an equilibrium state, therefore,

$$h(V) - h(V_r) = v\rho\;T\,(ds/dV)(V - V_r)$$

or, since the heat flux vector is zero in an equilibrium state, V_r,

$$h(V) = \rho v T\,ds/(ds/dV)(V - V_r)\,. \tag{17}$$

The mechanical dissipation can also be approximated by the first term of a series expansion. Thus,

$$\delta = -(\sigma - P)\dot{V}\,,$$
$$\delta(V) = -[d\sigma/dV - dP/dV]_{V_r}\,(V - V_r)\dot{V}. \tag{18}$$

We now note that the path in the $\sigma - V$ plane followed by a mass element is represented by the straight line joining the equilibrium end states (Rayleigh line.)[8] Its slope is given by

$$-j^2 = (P_1 - P_0)/(V_1 - V_0) = (\sigma - P_0)/(V - V_0), \tag{19}$$

where j is the mass flux,

$$j = \rho_0 v_0 = \rho_1 v_1.$$

(This relation can be verified by integrating the equations of continuity and motion through a steady shock transition and noting that it is the stress that enters into the equation of motion. Nothing in these two mechanical relations requires an assumption of thermodynamic equilibrium.) This relation allows us to write (18) as

$$\delta = (j^2 + dP/dV)(V - V_r)\dot{V}. \tag{20}$$

Returning to inequality (11), and using the expressions for γ, h, and δ from (14), (17), and (20), we have

$$\dot{V}(V - V_r)[j^2 + dP/dV + (dT/dV)(ds/dV)]_{V_r} \geq 0, \tag{21}$$

where, as previously stated, the derivatives are evaluated on the equilibrium surface along the path representing the projection of the real path.

There are four cases to consider depending on whether the shock is a compression or a rarefaction shock, and depending whether the reference equilibrium state is ahead of or behind the shock.

For compression shocks we have $\dot{V} < 0$, and,

(i) $V_r = V_1, \quad V > V_r$,

(ii) $V_r = V_0, \quad V < V_r$;

while for rarefaction shocks, $\dot{V} > 0$, and,

(iii) $V_r = V_1, \quad V < V_r$,

(iv) $V_r = V_0, \quad V > V_r$.

With respect to the sign of the bracketed quantity in (21) therefore, cases (i) and (iii) referring to the head of either type of shock are equivalent, as are cases (ii) and (iv) referring to the foot of the shock. Thus,

$$j^2 + dP/dV + (dT/dV)(ds/dV) \leq 0 \quad \text{(head)}, \tag{22a}$$

$$\geq 0 \quad \text{(foot)}. \tag{22b}$$

The directional derivatives of relation (21) can be expressed in terms of the derivatives of (3) and hence in terms of properties of the equilibrium surface by the identities,

$$\begin{aligned} ds/dV &= (\partial s/\partial V)_P + (\partial s/\partial P)_V (dP/dV) \\ &= (\partial P/\partial T)_s - (\partial V/\partial T)_s (dP/dV) \\ &= (\partial V/\partial T)_s [(\partial P/\partial V)_s - dP/dV] \end{aligned}$$

and

$$\begin{aligned} dT/dV &= (\partial T/\partial V)_s + (\partial T/\partial s)_V (ds/dV) \\ &= (\partial T/\partial V)_s + (\partial T/\partial s)_V (\partial V/\partial T)_s [(\partial P/\partial V)_s - dP/dV]. \end{aligned}$$

Substituting into (21) gives

$$\dot{V}(V - V_r)\{j^2 + (\partial P/\partial V)_s + (\partial V/\partial T)_s^2 (\partial T/\partial s)_V [(\partial P/\partial V)_s - dP/dV]^2\} \geq 0. \tag{23}$$

Although we make no direct use of it, we note that (23) can be written in a more symmetric way by use of the thermodynamic identity

$$(\partial V/\partial T)_s^2 (\partial T/\partial s)_V = [(\partial P/\partial V)_T - (\partial P/\partial V)_s]^{-1}.$$

Equation (23) then becomes

$$\dot{V}(V - V_r)\{[j^2 + (\partial P/\partial V)_s][(\partial P/\partial V)_T - (\partial P/\partial V)_s] + [(\partial P/\partial V)_s - dP/dV]^2\}_{V_r} \geq 0$$

or

$$\dot{V}(V - V_r)\{[j^2 + (\partial P/\partial V)_s][j^2 + (\partial P/\partial V)_T] - 2[j^2 + (\partial P/\partial V)_s] \times [j^2 + dP/dV] + [j^2 + dP/dV]^2\}_{V_r} \geq 0. \tag{24}$$

This relation is equivalent to (21) and is an expression of the second law, (11), to the first nonvanishing terms.

From (23) it is already clear that at the head of the shock, $\dot{V}(V - V_1) \leq 0$, we must have

$$j^2 + (\partial P/\partial V)_s \leq 0$$

or, in view of (3a),

$$-j^2(\partial V/\partial P)_s \leq 1. \tag{25}$$

It is readily shown, however, that $M^2 = -j^2(\partial V/\partial P)_s$ and therefore, (25) gives

$$M_1^2 \leq 1, \quad M_1 \leq 1 \tag{26}$$

as expected.

Examination of (23) or (24), at the foot of the shock, however, does not lead to the other expected inequality, i.e., $M_0^2 \geq 1$. Moreover, as Truesdell notes, invoking only the inequality (11), admits the possibility that either contribution to the entropy production could by itself be negative if it were compensated for by sufficient entropy production by the other term.[7] This would admit such peculiar circumstances as negative dissipation accompanying a large heat flux. Conversely, sufficiently large dissipation could be accompanied by heat flow in the same direction as the temperature gradient, i.e., heat could flow uphill.

It will now be assumed that each term of the inequality (11) must be positive. This appears justifiable, for example, by noting that according to the usual classical assumption (Fourier conduction law)

$$\mathbf{h} = -k\boldsymbol{\gamma},$$

where k is a positive coefficient. Under this law, the second term of the inequality (11) becomes

$$-\mathbf{h} \cdot \boldsymbol{\gamma} = k(\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}) > 0.$$

Moreover, it is usually assumed, in accordance with the Navier–Stokes equations (Ref. 4, p. 337), that the dissipative stress, $\sigma - P$, is, for one-dimensional flow,

$$\begin{aligned} \sigma - P &= -[(4/3)n + \xi](dv/dx) \\ &= -[(4/3)n + \xi]\rho\dot{V}, \end{aligned}$$

where the viscosity coefficients, n and ξ, are both negative. From Eq. (10), it then follows that under this assumption, the first term of the inequality (11) will be,

$$\rho\delta = [(4/3)n + \xi]\rho^2\dot{V}^2 > 0.$$

These assumptions concerning the constitutive relations can be considered to be empirical laws, or to be simply the first-order terms in a series expansion valid for small temperature or velocity gradients. The principal significance for our purposes is that there are no cross-coupling terms that could cause either term of inequality (11) to make a negative contribution to the entropy production.

In place of inequality (23), we now have two inequalities,

$$\mathring{V}(V-V_r)[j^2+dP/dV]\geqq 0 \tag{27a}$$

and

$$\mathring{V}(V-V_r)[(\partial P/\partial V)_s-dP/dV][(\partial P/\partial V)_T-dP/dV]\geqq 0. \tag{27b}$$

First considering the head of the shock, for which $\mathring{V}(V-V_r)\leqq 0$, we have

$$j^2+dP/dV\leqq 0;\quad dP/dV\leqq -j^2, \tag{28a}$$

$$[(\partial P/\partial V)_s-dP/dV][(\partial P/\partial V)_T-dP/dV]\leqq 0. \tag{28b}$$

The latter inequality requires that the slope, dP/dV, be intermediate between the isentropic and isothermal derivatives, while (28a) requires that it be less than the slope of the Rayleigh line. Consequently, the more restrictive of the inequalities

$$(\partial P/\partial V)_s\leqq dP/dV\leqq\begin{cases}-j^2\\(\partial P/\partial V)_T\end{cases} \tag{29}$$

obtains.

If we consider the foot of the shock, $\mathring{V}(V-V_0)\geqq 0$, we have, in place of (28),

$$j^2+dP/dV\geqq 0 \tag{30}$$

and

$$[(\partial P/\partial V)_s-dP/dV][(\partial P/\partial V)_T-dP/dV]\geqq 0. \tag{31}$$

These have the solutions,

$$dP/dV\geqq -j^2 \tag{32}$$

and either (i)

$$dP/dV\leqq(\partial P/\partial V)_s,\quad dP/dV\leqq(\partial P/\partial V)_T, \tag{33}$$

or, (ii)

$$dP/dV\geqq(\partial P/\partial V)_s,\quad dP/dV\geqq(\partial P/\partial V)_T. \tag{34}$$

The two solutions, (33) and (34), are seen to exclude the function $P(V)$ from the region between the isentrope and isotherm. If we assume (34) can be correct, however, and compare two hypothetical materials which differ only in the coefficient of thermal conduction, k, then the effect of heat flow would be to *increase* the mechanical dissipation. This is contrary to the Le Chatelier–Braun principle which states that secondary processes induced as a result of a primary process will act in a direction to reduce the primary thermodynamic stress difference.[9] Consequently, we take (33) to be the correct result, and it then follows that

$$-j^2\leqq dP/dV\leqq(\partial P/\partial V)_s \tag{35}$$

and, further,

$$-j^2(\partial V/\partial P)_s=M_0^2\geqq 1,\quad M_0\geqq 1. \tag{36}$$

This is the other result expected.

The restrictions on the slope dP/dV specified by (35) and by (29) are shown in Fig. 2.

It is clear from the diagram that an alternate argument for the exclusion of the solution (34) can be based on the continuity of the curve, $P(V)$. For weak shocks state 1 approaches state 0 and s_1 approaches s_0 (the so-called weak shock approximation.)[10] However, since $P(V)$ is excluded from the region between s_0 and T_0 and is confined between s_1 and T_1, $P(V)$ can be continuous only under the conditions shown, i.e., only if (34) is excluded.

It is interesting to note that the result (29) requires that, whenever the isotherm falls below the Rayleigh line at the head of the shock, i.e.,

$$(\partial P/\partial V)_T<-j^2,$$

then since $dP/dV<(\partial P/\partial V)_T$, it is necessary that some dissipation occur to account for at least the difference in stress between the isotherm and the Rayleigh line. Thus, under these conditions it is not possible to assume strictly nonviscous behavior no matter how conductive the material.[11] The converse is not true, however; there seem to be no restrictions that would rule out nonconducting but viscous behavior as assumed by

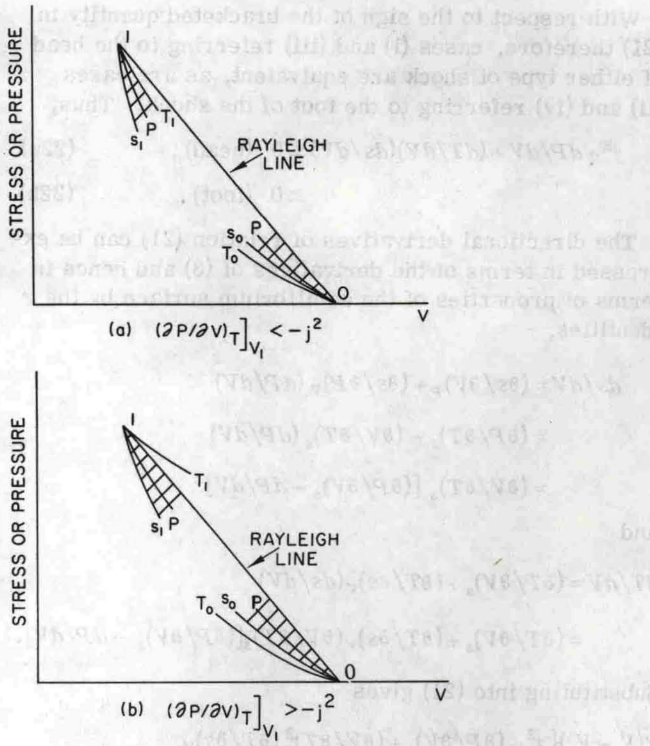

FIG. 2. Allowed regions for the slope dP/dV in the stress-volume plane for a compressive shock. Curves labeled s_0 and T_0 are the isentrope and isotherm through the initial state and those labeled s_1 and T_1 are corresponding curves through the final state. Curves labeled P are the projections of the path on the equilibrium surface traced by a mass element traversing the shock layer. The cross-hatched areas are the permissible bounds for P. (a) $(\partial P/\partial V)_T]_{V_1}<-j^2$; (b) $(\partial P/\partial V)_T]_{V_1}>-j^2$.

Band in his studies of shock structure.[8] For this latter case it is clear from (17) that $ds/dV=0$ and, moreover,

$$dP/dV=(\partial P/\partial V)_s+(\partial P/\partial s)_V(ds/dV)=(\partial P/\partial V)_s.$$

IV. CONCLUSIONS

We may summarize the conclusions as follows.

(1) The subsonic-supersonic conditions for shocks, $M_0 \geqq 1$, $M_1 \leqq 1$ are a consequence of the second law of thermodynamics for viscous, heat conducting fluids with arbitrary equation of state. It is not necessary to invoke the additional conditions,

$$(\partial^2 P/\partial V^2)_s > 0, \quad (\partial P/\partial s)_V > 0.$$

(2) The effect of heat conduction is to reduce the mechanical dissipation in accord with the Le Chatelier–Braun principle.

(3) Under the condition,

$$(\partial P/\partial V)_T < -j^2$$

at the head of the shock, it is not thermodynamically permissible to assume for any material that entropy production is due to heat conduction alone. That is, some viscous dissipation is necessary. This would rule out such predicted phenomena as the "isothermal discontinuity (Ref. 4, p. 342)."

(4) For all materials for which the quantity $(\partial P/\partial s)_V$ is positive in the shocked state, the entropy must attain a maximum value in the transition region. This conclusion agrees with that derived earlier for an ideal gas.[5]

These conclusions depend on the correctness of the assumption that the contributions to entropy production due to mechanical dissipation and to heat conduction are individually positive.

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